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ON DELTA ALPHA DERIVATIVE ON TIME SCALES

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Abstract. In this paper, we define and study the delta alpha derivative on time scales. Many basic properties of delta alpha derivative will be obtained.

1. Introduction

The theory of fractional calculus, which deals with the investigation and applications of derivatives and integrals of arbitrary order has a long history. The theory of fractional calculus developed mainly as a pure theoretical field of mathematics, in the last decades it has been used in various fields as mechanics, physics, chemistry, control theory, etc.[1-4]. Fractional calculus has undergone expanded study in recent years as a considerable interest both in mathematics and in applications[5-8, 19].

Recently, the authors in [9] define a new well-behaved simple fractional derivative called the conformable fractional derivative depending just on the basic limit definition of the derivative. In this paper we define the delta alpha derivative on time scales, which give a common generalization of the conformable fractional derivative and the usual delta derivative [10-11].

The theory of time scales was introduced for the first time in 1988 by Hilger [12] to unify the theory of difference equations and the theory of differential equations. It has been extensively studied on various aspects by several authors [13-18].

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2. Preliminaries

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma(t)$ by $\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}$ where $\inf \emptyset = \sup\{\mathbb{T}\}$, while the backward jump operator $\rho(t)$ is defined by $\rho(t) = \sup\{s < t : s \in \mathbb{T}\}$ where $\sup \emptyset = \inf\{\mathbb{T}\}$.

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that t is left-dense. A point $t \in \mathbb{T}$ is dense if it is right and left dense; isolated if it is right and left scattered. The forward graininess function $\mu(t)$ and the backward graininess function $\eta(t)$ are defined by $\mu(t) = \sigma(t) - t$, $\eta(t) = t - \rho(t)$ for all $t \in \mathbb{T}$ respectively. If $\sup \mathbb{T}$ is finite and left-scattered, then we define $\mathbb{T}^k := \mathbb{T} \backslash \sup \mathbb{T}$, otherwise $\mathbb{T}^k := \mathbb{T}$; if $\inf \mathbb{T}$ is finite and right-scattered, then $\mathbb{T}_k := \mathbb{T} \backslash \inf \mathbb{T}$, otherwise $\mathbb{T}_k := \mathbb{T}$. We set $\mathbb{T}_k^k := \mathbb{T}^k \bigcap \mathbb{T}_k$.

A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is called regulated provided its right-sided limits exist at all right-dense point of $[a,b)_{\mathbb{T}}$ and its left-sided limits exist at all left-dense point of $(a,b]_{\mathbb{T}}$. Throughout we will frequently write $f^{\sigma}(t) = f(\sigma(t))$.

A function $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^k$ if there exists a number $f^{\Delta}(t)$ such that, for each $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|f^{\sigma}(t) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s|$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta derivative of f at t and we say that f is delta differentiable if f is delta differentiable for all $t \in \mathbb{T}^k$. Throughout this paper, $\alpha \in (0,1]$.

3. The delta alpha derivative

Definition 3.1. Let $\mathbb{T}$ be a time scale and $\alpha \in (0,1]$. A function $f : \mathbb{T} \to \mathbb{R}$ is $\Delta_{\alpha}-$differentiable at $t \in \mathbb{T}^k$ if there exists a number $\mathbf{T}_{\alpha}(f^{\Delta})(t)$ such that, for each $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|(f^{\sigma}(t) - f(s))\sigma(t)^{1-\alpha} - \mathbf{T}_{\alpha}(f^{\Delta})(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s|$$

for all $s \in U$. We call $\mathbf{T}_{\alpha}(f^{\Delta})(t)$ the $\Delta_{\alpha}-$derivative of f at t and we say that f is $\Delta_{\alpha}-$differentiable if f is $\Delta_{\alpha}-$differentiable for all $t \in \mathbb{T}^k$.

Some useful properties of the $\Delta_\alpha-$derivative of f of order α are given in the next theorem.

THEOREM 3.2. *Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}^k$ and $\alpha \in (0,1]$. Then we have the following:*

(i) *If f is $\Delta_\alpha-$differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is $\Delta_\alpha-$differentiable at t with*

$$\mathbf{T}_\alpha(f^\Delta)(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}\sigma(t)^{1-\alpha}.$$

(iii) *If t is right-dense, then f is $\Delta_\alpha-$differentiable at t if and only if the limit*

$$\lim_{s\to t} \frac{f(t) - f(s)}{t - s} t^{1-\alpha}$$

exists as a finite number. In this case,

$$\mathbf{T}_\alpha(f^\Delta)(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s} t^{1-\alpha}.$$

(iv) *If f is $\Delta_\alpha-$differentiable at t, then*

$$f^\sigma(t) = f(t) + \mu(t)\mathbf{T}_\alpha(f^\Delta)(t)\sigma(t)^{\alpha-1}.$$

Proof. Part (i). Assume that f is $\Delta_\alpha-$differentiable at t, then for each $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|(f^\sigma(t) - f(s))\sigma(t)^{1-\alpha} - \mathbf{T}_\alpha(f^\Delta)(t)(\sigma(t) - s)| \le \epsilon^*|\sigma(t) - s|$$

for all $s \in U$, here

$$\epsilon^* = \frac{\epsilon}{1 + |\mathbf{T}_\alpha(f^\Delta)(t)| + 2\mu(t)}|\sigma(t)^{1-\alpha}|.$$

Then for all $s \in U \cap (t - \epsilon^*, t + \epsilon^*)$, we have that

$$\begin{aligned}
&|f(t) - f(s)| \\
&= |f^\sigma(t) - f(s) - \mathbf{T}_\alpha(f^\Delta)(t)(\sigma(t) - s)\sigma(t)^{\alpha-1} \\
&\quad - \left\{f^\sigma(t) - f(t) - \mathbf{T}_\alpha(f^\Delta)(t)(\sigma(t) - t)\sigma(t)^{\alpha-1}\right\} + \mathbf{T}_\alpha(f^\Delta)(t)(t - s)\sigma(t)^{\alpha-1}| \\
&\le \epsilon^*|(\sigma(t) - s)\sigma(t)^{\alpha-1}| + \epsilon^*|(\sigma(t) - t)\sigma(t)^{\alpha-1}| + |\mathbf{T}_\alpha(f^\Delta)(t)(t - s)\sigma(t)^{\alpha-1}| \\
&\le \epsilon^*|\sigma(t)^{\alpha-1}|\left(\mu(t) + |s - t| + \mu(t) + |\mathbf{T}_\alpha(f^\Delta)(t)|\right) \\
&< \epsilon^*|\sigma(t)^{\alpha-1}|\left(1 + |\mathbf{T}_\alpha(f^\Delta)(t)| + 2\mu(t)\right) \\
&= \epsilon.
\end{aligned}$$

It follows that f is continuous at t.

Part (ii). Assume that f is continuous at t and t is right-scattered, By continuity,

$$\lim_{s\to t}\frac{f^\sigma(t)-f(s)}{\sigma(t)-s}\sigma(t)^{1-\alpha}=\frac{f^\sigma(t)-f(t)}{\sigma(t)-t}\sigma(t)^{1-\alpha}=\frac{f^\sigma(t)-f(s)}{\mu(t)}\sigma(t)^{1-\alpha}.$$

Hence, given $\varepsilon>0$, there exists a neighborhood U of t such that

$$\left|\frac{f^\sigma(t)-f(s)}{\sigma(t)-s}\sigma(t)^{1-\alpha}-\frac{f^\sigma(t)-f(s)}{\mu(t)}\sigma(t)^{1-\alpha}\right|\le\epsilon$$

for all $s\in U$. It follows that

$$\left|(f^\sigma(t)-f(s))\,\sigma(t)^{1-\alpha}-\frac{f^\sigma(t)-f(s)}{\mu(t)}\,(\sigma(t)-s)\,\sigma(t)^{1-\alpha}\right|\le\epsilon|(\sigma(t)-s)|$$

for all $s\in U$. Hence we get the desired result

$$\mathbf{T}_\alpha(f^\Delta)(t)=\frac{f^\sigma(t)-f(t)}{\mu(t)}\sigma(t)^{1-\alpha}.$$

Part (iii). Assume that f is $\Delta_\alpha-$differentiable at t and t is right-dense. Then for each $\varepsilon>0$, there exists a neighborhood U of t such that

$$|(f^\sigma(t)-f(s))\sigma(t)^{1-\alpha}-\mathbf{T}_\alpha(f^\Delta)(t)(\sigma(t)-s)|\le\epsilon^*|(\sigma(t)-s)|$$

for all $s\in U$. Since $\sigma(t)=t$ we have that

$$|(f(t)-f(s))t^{1-\alpha}-\mathbf{T}_\alpha(f^\Delta)(t)(t-s)|\le\epsilon^*|(t-s)|$$

for all $s\in U$. It follows that

$$\left|\frac{f(t)-f(s)}{t-s}t^{1-\alpha}-\mathbf{T}_\alpha(f^\Delta)(t)\right|\le\epsilon$$

for all $s\in U$, $s\neq t$. Hence we get the desired result

$$\mathbf{T}_\alpha(f^\Delta)(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}t^{1-\alpha}.$$

On the other hand, if the limit

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}t^{1-\alpha}$$

exists as a finite number and is equal to J, then for each $\varepsilon>0$, there exists a neighborhood U of t such that

$$|(f(t)-f(s))t^{1-\alpha}-J(t-s)|\le\epsilon|(t-s)|$$

for all $s\in U$. Since t is right-dense, $\sigma(t)=t$, we have that

$$|(f^\sigma(t)-f(s))\sigma(t)^{1-\alpha}-J(\sigma(t)-s)|\le\epsilon|(\sigma(t)-s)|.$$

Hence, f is $\Delta_\alpha-$differentiable at t and

$$\mathbf{T}_\alpha(f^\Delta)(t) = \lim_{s\to t}\frac{f(t)-f(s)}{t-s}t^{1-\alpha}.$$

Part (iv). If t is right-dense, then $\mu(t)=0$ and we have that

$$f^\sigma(t) = f(t) = f(t) + \mu(t)\mathbf{T}_\alpha(f^\Delta)(t)\sigma(t)^{\alpha-1}.$$

If t is right-scattered, then $\sigma(t) > t$, then by (ii)

$$f^\sigma(t) = f(t) + \mu(t)\frac{f^\sigma(t)-f(t)}{\mu(t)} = f(t) + \mu(t)\mathbf{T}_\alpha(f^\Delta)(t)\sigma(t)^{\alpha-1}.$$

□

Corollary 3.3. *Again we consider the two cases $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$.*

(i) *If $\mathbb{T}=\mathbb{R}$, then $f:\mathbb{R}\to\mathbb{R}$ is $\Delta_\alpha-$differentiable at $t\in\mathbb{R}$ if and only if the limit*

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}t^{1-\alpha}$$

exists as a finite number. In this case,

$$\mathbf{T}_\alpha(f^\Delta)(t) = \lim_{s\to t}\frac{f(t)-f(s)}{t-s}t^{1-\alpha}.$$

If $\alpha=1$, then we have that

$$\mathbf{T}_\alpha(f^\Delta)(t) = f^\Delta(t) = f'(t).$$

(ii) *If $\mathbb{T}=\mathbb{Z}$, then $f:\mathbb{Z}\to\mathbb{R}$ is $\Delta_\alpha-$differentiable at $t\in\mathbb{Z}$ with*

$$\mathbf{T}_\alpha(f^\Delta)(t) = \frac{f(t+1)-f(t)}{1}(t+1)^{1-\alpha} = (t+1)^{1-\alpha}\left(f(t+1)-f(t)\right)$$

If $\alpha=1$, then we have that

$$\mathbf{T}_\alpha(f^\Delta)(t) = f(t+1)-f(t) = \Delta f(t),$$

where Δ is the usual forward difference operator.

Example 3.4. (i) *If $f:\mathbb{T}\to\mathbb{R}$ is defined by $f(t)=C$ for all $t\in\mathbb{T}$, where $C\in\mathbb{R}$ is constant, then*

$$\mathbf{T}_\alpha(f^\Delta)(t)\equiv 0.$$

This is because for any $\epsilon>0$,

$$|(f^\sigma(t)-f(s))\sigma(t)^{1-\alpha} - 0\cdot(\sigma(t)-s)| = |C-C| = 0 \le \epsilon|(\sigma(t)-s)|$$

holda for all $s\in\mathbb{T}$.

(ii) *If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then*

$$\mathbf{T}_\alpha(f^\Delta)(t) = \sigma(t)^{1-\alpha}.$$

This is because for any $\epsilon > 0$,

$$\begin{aligned} & |(f^\sigma(t) - f(s))\sigma(t)^{1-\alpha} - \sigma(t)^{1-\alpha} \cdot (\sigma(t) - s)| \\ = \; & |(\sigma(t) - s)\sigma(t)^{1-\alpha} - \sigma(t)^{1-\alpha} \cdot (\sigma(t) - s)| \\ = \; & 0 \leq \epsilon|(\sigma(t) - s)|. \end{aligned}$$

holda for all $s \in \mathbb{T}$.

If $\alpha = 1$, then $\mathbf{T}_\alpha(f^\Delta)(t) \equiv 1$.

Example 3.5. *If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t^2$ for all $t \in \mathbb{T} := \{\frac{n}{2} : n \in \mathbb{N}_0\}$, then from Theorem 3.2 (ii) we have that f is $\Delta_\alpha-$differentiable at $t \in \mathbb{T}$ with*

$$\mathbf{T}_\alpha(f^\Delta)(t) = \left(2t + \frac{1}{2})(t + \frac{1}{2}\right)^{1-\alpha}.$$

Theorem 3.6. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are $\Delta_\alpha-$differentiable at $t \in \mathbb{T}^k$. Then:*

(i) *for any constant λ_1, λ_2, the sum $\lambda_1 f + \lambda_2 g : \mathbb{T} \to \mathbb{R}$ is $\Delta_\alpha-$differentiable at $t \in \mathbb{T}^k$ with*

$$\mathbf{T}_\alpha((\lambda_1 f + \lambda_2 g)^\Delta)(t) = \lambda_1 \mathbf{T}_\alpha(f^\Delta)(t) + \lambda_2 \mathbf{T}_\alpha(g^\Delta)(t).$$

(ii) *If f and g are continuous, then the product $fg : \mathbb{T} \to \mathbb{R}$ is $\Delta_\alpha-$differentiable at t with*

$$\begin{aligned} \mathbf{T}_\alpha(fg)^\Delta(t) &= \mathbf{T}_\alpha(f^\Delta)(t)g(t) + f^\sigma(t)\mathbf{T}_\alpha(g^\Delta)(t) \\ &= f(t)\mathbf{T}_\alpha(g^\Delta)(t) + \mathbf{T}_\alpha(f^\Delta)(t)g^\sigma(t). \end{aligned}$$

(iii) *If $f(t)f^\sigma(t) \neq 0$, then $\frac{1}{f}$ is $\Delta_\alpha-$differentiable at t with*

$$\mathbf{T}_\alpha\left(\frac{1}{f}\right)^\Delta(t) = -\frac{\mathbf{T}_\alpha(f^\Delta)(t)}{f(t)f^\sigma(t)}.$$

(iv) *If $g(t)g^\sigma(t) \neq 0$, then $\frac{f}{g}$ is $\Delta_\alpha-$differentiable at t with*

$$\mathbf{T}_\alpha\left(\frac{f}{g}\right)^\Delta(t) = \frac{\mathbf{T}_\alpha(f^\Delta)(t)g(t) - f(t)\mathbf{T}_\alpha(g^\Delta)(t)}{g(t)g^\sigma(t)}.$$

Proof. Part (i). Let $\varepsilon > 0$. Then there exist neighborhoods U_1 and U_2 of t with

$$|(\lambda_1 f^\sigma(t) - \lambda_1 f(s))\sigma(t)^{1-\alpha} - \lambda_1 \mathbf{T}_\alpha(f^\Delta)(t)(\sigma(t) - s)| \leq \frac{\epsilon}{2}|\lambda_1(\sigma(t) - s)|$$

for all $s \in U_1$ and

$$|(\lambda_2 g^\sigma(t) - \lambda_2 g(s))\sigma(t)^{1-\alpha} - \lambda_2 \mathbf{T}_\alpha(g^\Delta)(t)(\sigma(t) - s)| \le \frac{\epsilon}{2}|\lambda_2(\sigma(t) - s)|$$

for all $s \in U_2$. Therefore, $\lambda_1 f$ and $\lambda_2 g$ are conformable fractional differentiable at t and $\mathbf{T}_\alpha((\lambda_1 f)^\Delta)(t) = \lambda_1 \mathbf{T}_\alpha(f^\Delta)(t)$, $\mathbf{T}_\alpha((\lambda_2 g)^\Delta)(t) = \lambda_2 \mathbf{T}_\alpha(g^\Delta)(t)$ holds at t.

Let $U = U_1 \cap U_2$, $\lambda = \max\{\lambda_1, \lambda_2\}$. Then we have for all $s \in U$

$$\begin{aligned}
&|((\lambda_1 f^\sigma + \lambda_2 g^\sigma)(t) - (\lambda_1 f + \lambda_2 g)(s))\sigma(t)^{1-\alpha} \\
&\quad - (\lambda_1 \mathbf{T}_\alpha(f^\Delta)(t) + \lambda_2 \mathbf{T}_\alpha(g^\Delta)(t))(\sigma(t) - s)| \\
&\le |(\lambda_1 f^\sigma(t) - \lambda_1 f(s))\sigma(t)^{1-\alpha} - \lambda_1 \mathbf{T}_\alpha(f^\Delta)(t)(\sigma(t) - s)| \\
&\quad + |(\lambda_2 g^\sigma(t) - \lambda_2 g(s))\sigma(t)^{1-\alpha} - \lambda_2 \mathbf{T}_\alpha(g^\Delta)(t)(\sigma(t) - s)| \\
&\le \frac{\epsilon}{2}|\lambda_1(\sigma(t) - s)| + \frac{\epsilon}{2}|\lambda_2(\sigma(t) - s)| \le \epsilon|\lambda(\sigma(t) - s)|.
\end{aligned}$$

Therefore $\lambda_1 f + \lambda_2 g$ is $\Delta_\alpha-$differentiable at $t \in \mathbb{T}^k$ with

$$\mathbf{T}_\alpha((\lambda_1 f + \lambda_2 g)^\Delta)(t) = \lambda_1 \mathbf{T}_\alpha(f^\Delta)(t) + \lambda_2 \mathbf{T}_\alpha(g^\Delta)(t).$$

Part (ii). Let $0 < \varepsilon < 1$. Define

$$\epsilon^* = \frac{\epsilon}{1 + |g^\sigma(t)| + |f(t)| + |\mathbf{T}_\alpha(g^\Delta)(t)|},$$

then $0 < \epsilon^* < 1$. $f, g : \mathbb{T} \to \mathbb{R}$ are $\Delta_\alpha-$differentiable at $t \in \mathbb{T}^k$. Then there exists neighborhoods U_1 and U_2 of t with

$$|(f^\sigma(t) - f(s))\sigma(t)^{1-\alpha} - \mathbf{T}_\alpha(f^\Delta)(t)(\sigma(t) - s)| \le \epsilon^*|\sigma(t) - s|$$

for all $s \in U_1$ and

$$|g^\sigma(t) - g(s))\sigma(t)^{1-\alpha} - \mathbf{T}_\alpha(g^\Delta)(t)(\sigma(t) - s)| \le \epsilon^*|\sigma(t) - s|$$

for all $s \in U_2$.

From Theorem 3.2 (i), there exists neighborhoods U_3 of t with

$$|f(t) - f(s)| \le \epsilon^*$$

for all $s \in U_3$.

Let $U = U_1 \cap U_2 \cap U_3$. Then we have for all $s \in U$

$$\begin{aligned}
&|[f^\sigma(t)g^\sigma(t) - f(s)g(s)]\sigma(t)^{1-\alpha} - [\mathbf{T}_\alpha(f^\Delta)(t)g^\sigma(t) \\
&\quad + f(t)\mathbf{T}_\alpha(g^\Delta)(t)](\sigma(t) - s)| \\
&\le |[(f^\sigma(t) - f(s))\sigma(t)^{1-\alpha} - \mathbf{T}_\alpha(f^\Delta)(t)(\sigma(t) - s)]g^\sigma(t)| \\
&\quad + |[(g^\sigma(t) - g(s))\sigma(t)^{1-\alpha} - \mathbf{T}_\alpha(g^\Delta)(t)(\sigma(t) - s)]f(t)| \\
&\quad + |[(g^\sigma(t) - g(s))\sigma(t)^{1-\alpha} - \mathbf{T}_\alpha(g^\Delta)(t)(\sigma(t) - s)](f(s) - f(t))| \\
&\quad + |\mathbf{T}_\alpha(g^\Delta)(t)(\sigma(t) - s)](f(s) - f(t))| \\
&\le \epsilon^*|\sigma(t) - s| \cdot \left(|g^\sigma(t)| + |f(t)| + \epsilon^* + |\mathbf{T}_\alpha(g^\Delta)(t)|\right). \\
&\le \epsilon|\sigma(t) - s|.
\end{aligned}$$

Thus

$$\mathbf{T}_\alpha(fg)^\Delta(t) = f(t)\mathbf{T}_\alpha(g^\Delta)(t) + \mathbf{T}_\alpha(f^\Delta)(t)g^\sigma(t).$$

The other product rule formula follows by interchanging the role of functions f and g.

Part (iii). From Example 3.4 we have that

$$\mathbf{T}_\alpha\left(f \cdot \frac{1}{f}\right)^\Delta(t) = \mathbf{T}_\alpha(1)^\Delta(t) = 0.$$

Therefore,

$$\mathbf{T}_\alpha\left(\frac{1}{f}\right)^\Delta(t)f^\sigma(t) + \mathbf{T}_\alpha(f^\Delta)(t)\frac{1}{f(t)} = 0$$

and consequently

$$\mathbf{T}_\alpha\left(\frac{1}{f}\right)^\Delta(t) = -\frac{\mathbf{T}_\alpha(f^\Delta)(t)}{f(t)f^\sigma(t)}.$$

Part (iv). We use (ii) and (iii) to calculate

$$\begin{aligned}
\mathbf{T}_\alpha\left(\frac{f}{g}\right)^\Delta(t) &= f(t)\mathbf{T}_\alpha\left(\frac{1}{g}\right)^\Delta(t) + \mathbf{T}_\alpha(f^\Delta)(t)\frac{1}{g^\sigma(t)} \\
&= -f(t)\frac{\mathbf{T}_\alpha(g^\Delta)(t)}{g(t)g^\sigma(t)} + \mathbf{T}_\alpha(f^\Delta)(t)\frac{1}{g^\sigma(t)} \\
&= \frac{\mathbf{T}_\alpha(f^\Delta)(t)g(t) - f(t)\mathbf{T}_\alpha(g^\Delta)(t)}{g(t)g^\sigma(t)}.
\end{aligned}$$

□

Corollary 3.7. *Let $\mathbb{T}$ be a time scale, c be a constant and $m \in \mathbb{N}$.*
(i) For f defined by $f(t) = (t-c)^m$ we have that

$$\mathbf{T}_\alpha(f^\Delta)(t) = \sigma(t)^{1-\alpha} \sum_{i=0}^{m-1} (\sigma(t)-c)^i (t-c)^{m-1-i}.$$

(ii) For g defined by $g(t) = \frac{1}{(t-c)^m}$ we have that

$$\mathbf{T}_\alpha(g^\Delta)(t) = -\sigma(t)^{1-\alpha} \sum_{i=0}^{m-1} \frac{1}{(\sigma(t)-c)^{m-i}(t-c)^{i+1}}$$

provided $(\sigma(t)-c)(t-c) \neq 0$.

Proof. Part (i). We prove the first formula by induction. If $m = 1$, then $f(t) = t - c$, and clearly $\mathbf{T}_\alpha(f^\Delta)(t) = \sigma(t)^{1-\alpha}$ holds by Example 3.4 and Theorem 3.6 (i). Now we assume that

$$\mathbf{T}_\alpha(f^\Delta)(t) = \sigma(t)^{1-\alpha} \sum_{i=0}^{m-1} (\sigma(t)-c)^i (t-c)^{m-1-i}$$

holds for $f(t) = (t-c)^m$ and let $F(t) = (t-c)^{m+1} = (t-c)f(t)$. We use Theorem 3.6 (ii) to obtain

$$\begin{aligned}
\mathbf{T}_\alpha(F)^\Delta(t) &= \sigma(t)^{1-\alpha} f^\sigma(t) + (t-c)\mathbf{T}_\alpha(f^\Delta)(t) \\
&= \sigma(t)^{1-\alpha}(\sigma(t)-c)^m + (t-c)\sigma(t)^{1-\alpha} \sum_{i=0}^{m-1} (\sigma(t)-c)^i (t-c)^{m-1-i} \\
&= \sigma(t)^{1-\alpha} \sum_{i=0}^{m} (\sigma(t)-c)^i (t-c)^{m-i}.
\end{aligned}$$

Hence, part (i) holds.

Part (ii). For $g(t) = \frac{1}{(t-c)^m}$ we use Theorem 3.6 (iii) to obtain

$$\begin{aligned}
\mathbf{T}_\alpha(g)^\Delta(t) &= -\frac{\mathbf{T}_\alpha(f^\Delta)(t)}{f(t)f^\sigma(t)} \\
&= -\frac{\sigma(t)^{1-\alpha} \sum_{i=0}^{m-1} (\sigma(t)-c)^i (t-c)^{m-1-i}}{(\sigma(t)-c)^m (t-c)^m} \\
&= -\sigma(t)^{1-\alpha} \sum_{i=0}^{m-1} \frac{1}{(\sigma(t)-c)^{m-i}(t-c)^{i+1}}
\end{aligned}$$

provided $(\sigma(t)-c)(t-c) \neq 0$. $\square$

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